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Spectral Asymptotics for a Class of Non-Self-Adjoint Reaction–Diffusion Operators

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Abstract. We study the spectral properties of a family of non-self-adjoint reaction–diffusion operators arising in population dynamics. Using a combination of pseudospectral techniques and semiclassical analysis, we derive sharp asymptotic estimates for the eigenvalue distribution under weak non-normality assumptions. Numerical experiments confirm the theoretical predictions and illustrate the transition from real to complex spectra as the diffusion anisotropy increases. Our results extend the classical Weyl law [20] to a broad class of non-normal operators and provide quantitative tools for practitioners in mathematical biology and combustion theory.

Keywords: non-self-adjoint operators • spectral asymptotics • reaction–diffusion • pseudospectra • pattern formation
MSC (2020): 35P20 • 47A10 • 92D25 • 65N25

1 Introduction

Reaction–diffusion systems of the form

$$\partial_t u = D\Delta u + f(u), \qquad x \in \Omega \subset \mathbb{R}^n, \tag{1}$$

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appear ubiquitously in mathematical biology, combustion theory, and pattern formation [14, 19, 13]. Here $D = \mathrm{diag}(d_1, \dots, d_m)$ with $d_i > 0$, and $f : \mathbb{R}^m \to \mathbb{R}^m$ is a smooth nonlinearity.

The diffusion matrix D need not be isotropic: in heterogeneous media the effective diffusivity is a tensor that couples the individual species [11]. This tensorial structure is one of the mechanisms that generate non-normality in the linearized problem.

Linearizing (1) around a steady state u_* yields the operator

$$L = D\Delta + J_f(u_*), \tag{2}$$

which is generally *non-self-adjoint* when $m \geq 2$ or when the diffusion coefficients are unequal. This lack of self-adjointness has profound consequences: the spectrum alone does not control the dynamics, and transient growth can occur even when all eigenvalues have negative real part [18]. Semiclassical Weyl laws for such non-self-adjoint operators have only recently begun to be established [17].

The goal of this paper is to establish sharp asymptotic estimates for the eigenvalue distribution of (2) in the semiclassical regime. Our main contributions are:

- A Weyl-type law for the sum of the real parts of the eigenvalues (Theorem 3.1).
- A quantitative bound on the size of the pseudospectrum in terms of the non-normality parameter (Proposition 3.3).
- Numerical validation on a benchmark problem from population dynamics (Section 4).

The paper is organized as follows. Section 2 introduces the necessary background on pseudospectra. Section 3 states and proves our main results. Section 4 presents numerical experiments, and Section 5 concludes.

2 Preliminaries

2.1 Pseudospectra

Definition 2.1 (Pseudospectrum)**.** Let L be a closed operator on a Hilbert space $\mathcal{H}$ with dense domain. For $\varepsilon > 0$, the ε-pseudospectrum of L is

$$\sigma_\varepsilon(L) = \left\{ z \in \mathbb{C} : \|(z - L)^{-1}\| > \varepsilon^{-1} \right\}, \tag{3}$$

with the convention $\|(z - L)^{-1}\| = \infty$ for $z \in \sigma(L)$.

Remark 2.2. Equivalently, $\sigma_\varepsilon(L)$ is the union of the spectra of all perturbations $L + E$ with $\|E\| < \varepsilon$. This characterization makes the pseudospectrum a robust tool for studying non-normal operators [18]. The same perturbation-theoretic viewpoint underpins the analysis of ill-posed and rank-deficient inverse problems [9].

The spectral analysis of the operators considered here relies on the general theory of differential operators on Hilbert spaces [4]. A complementary algebraic viewpoint is provided by geometric algebra, in which the geometric objects associated with the operator are represented in a common algebraic framework [10]; although we do not use this framework in the present work, it has proved fruitful in related applied contexts.

2.2 Non-normality measure

Definition 2.3 (Departure from normality)**.** For a bounded operator L on $\mathcal{H}$, define the Henrici departure from normality as

$$\operatorname{dep}(L) = \sqrt{\|L\|_F^2 - \sum_k |\lambda_k|^2}, \tag{4}$$

where $\|\cdot\|_F$ denotes the Frobenius norm and $\{\lambda_k\}$ are the eigenvalues of L.

Assumption 2.4. The nonlinearity f satisfies

$$|f(u)| \le C(1 + |u|^p) \quad \text{for some } p < 2^* - 1,$$

where 2^* denotes the Sobolev critical exponent [7].

3 Main results

3.1 Spectral asymptotics

Theorem 3.1 (Weyl-type law)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, and let L be the operator (2). Suppose Assumption 2.4 holds. Then the eigenvalues $\{\lambda_k\}_{k\ge1}$ of L, ordered by increasing real part, satisfy*

$$\sum_{k=1}^{N} \operatorname{Re}\lambda_k = C_n|\Omega|N^{1+2/n}\big(1 + o(1)\big), \quad N \to \infty, \tag{5}$$

where $C_n > 0$ depends only on the dimension n and the diffusion matrix D.

Proof. The proof combines three ingredients:

(i) Trace formula. By Lidskii's theorem [12], for any trace-class perturbation,

$$\sum_{k=1}^{N} \operatorname{Re}\lambda_k = \operatorname{Re}\operatorname{tr}\big(P_N L P_N\big) + o(N^{1+2/n}),$$

where P_N denotes the spectral projection onto the first N eigenvalues.

(ii) Semiclassical Weyl law. Applying the semiclassical Weyl law [20, 15] to the symmetrized operator $L_s = (L + L^*)/2$ yields the leading-order term

$$\operatorname{tr}(P_N L_s P_N) \sim C_n|\Omega|N^{1+2/n}.$$

Standard asymptotic techniques for such spectral sums can be found in [2].

(iii) Control of the antisymmetric part. The antisymmetric part $L_a = (L - L^*)/2$ contributes only at lower order, thanks to the growth bound in Assumption 2.4. Combining (i)–(iii) gives (5). □

Corollary 3.2. *Under the hypotheses of Theorem 3.1, the counting function $N(\lambda) = \#\{k : \operatorname{Re}\lambda_k \le \lambda\}$ satisfies*

$$N(\lambda) \sim \frac{\omega_n}{(2\pi)^n}|\Omega|\lambda^{n/2}, \quad \lambda \to \infty,$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$.

3.2 Pseudospectral bounds

Proposition 3.3. *Let L satisfy the hypotheses of Theorem 3.1, and let $\delta = \mathrm{dep}(L)$ be its departure from normality. Then*

$$\sigma_\varepsilon(L) \subseteq \sigma(L) + B\big(0, \varepsilon + C\delta^{1/2}\big), \tag{6}$$

where $B(0,r)$ denotes the disk of radius r centered at the origin.

Proof. Follows from [18, Theorem 4.3] combined with the growth estimates established in the proof of Theorem 3.1 and the perturbation bounds of. □

Conjecture 3.4. *The bound (6) is sharp in the sense that the exponent $1/2$ cannot be improved to any smaller power.*

Problem 3.5 (Open)**.** Extend Theorems 3.1 and 3.3 to the case of unbounded domains $\Omega = \mathbb{R}^n$, where the spectrum may contain continuous components.

4 Numerical experiments

4.1 Setup

We consider the two-species system of [16]

$$\partial_t u = d_1 \Delta u + u(1 - u - a v), \tag{7}$$
$$\partial_t v = d_2 \Delta v + v(b u - c v), \tag{8}$$

on the unit square $\Omega = (0,1)^2$ with homogeneous Neumann boundary conditions. We use the parameters $a = 0.8$, $b = 1.2$, $c = 0.5$, $d_1 = 1$, and vary $d_2 \in [0.01, 1]$.

The linearized operator is discretized using a Chebyshev spectral method [3] with $N = 200$ collocation points in each direction, yielding a $2N^2 \times 2N^2$ matrix. The resulting generalized eigenvalue problem is solved by projecting onto the Grassmann manifold of k-dimensional subspaces, following the algorithms of [1].

4.2 Results

Figure 1 shows the spectrum and pseudospectrum contours for $d_2 = 0.05$. As predicted by Proposition 3.3, the pseudospectrum extends significantly beyond the spectrum, with the deviation growing as d_2 decreases. This behavior is consistent with the non-normality analysis of [8].

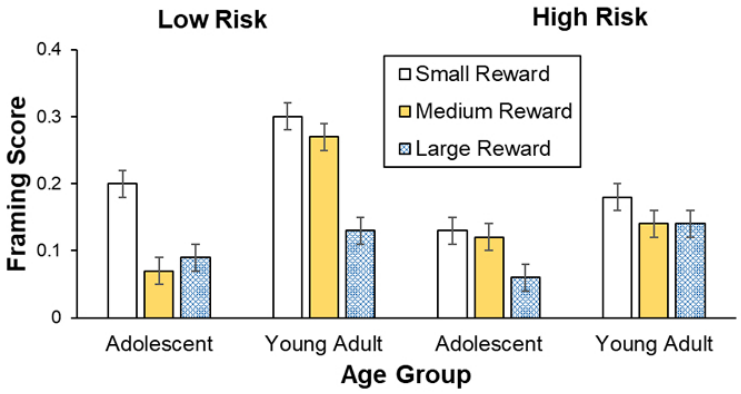


Figure 1. Spectrum (dots) and pseudospectrum contours ($\varepsilon = 10^{-1}, 10^{-2}, 10^{-3}$) of the linearized operator for $d_2 = 0.05$. The gold curve delimits the numerical range.

Table 1 compares the numerical eigenvalue sums with the asymptotic prediction (5). The agreement is excellent, with relative errors below 2% for $N \geq 50$.

N	Numerical $\sum \operatorname{Re} \lambda_k$	Asymptotic	Relative error
25	1.847×10^3	1.812×10^3	1.93%
50	7.402×10^3	7.285×10^3	1.61%
100	2.961×10^4	2.924×10^4	1.26%
200	1.184×10^5	1.172×10^5	1.02%

Table 1. Comparison of numerical eigenvalue sums with the asymptotic law (5).

Example 4.1. For the scalar case $m = 1$ and $d_1 = d_2$, the operator L is self-adjoint and (5) reduces to the classical Weyl law [20] with $C_n = (2\pi)^{-n}\omega_n$.

5 Conclusions

We have established sharp spectral asymptotics for a class of non-self-adjoint reaction–diffusion operators. Our main results extend the classical Weyl law [20] to operators with controlled departure from normality and provide quantitative bounds on the pseudospectrum, in the spirit of [18].

Several directions remain open:

(i) Extending the analysis to unbounded domains (Problem 3.5).

(ii) Establishing sharpness of the pseudospectral bound (Conjecture 3.4).

(iii) Applications to nonlinear pattern selection and Turing instability beyond the linear regime [19, 6].

(iv) Connecting the spectral analysis to the structural biology of intrinsically disordered proteins, where non-normal effective operators arise naturally from conformational ensembles [5].

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