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Spectral Asymptotics for a Class of Non-Self-Adjoint Reaction–Diffusion Operators

An Introduction to Pseudospectral Methods

María González, John A. Smith, and Klaus Weber

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Preface

This book provides a self-contained introduction to the spectral theory of non-self-adjoint reaction–diffusion operators, with a particular emphasis on pseudospectral techniques and their applications to mathematical biology and combustion theory.

The material is organized into four chapters. Chapter 1 motivates the problem and outlines the main contributions. Chapter 2 introduces the necessary background on pseudospectra and non-normality. Chapter 3 states and proves the main spectral asymptotics. Chapter 4 presents numerical experiments and discusses the transition from real to complex spectra.

The book is intended for graduate students and researchers in applied mathematics, with a working knowledge of functional analysis and partial differential equations.

The Authors
Spring 2026

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1 Introduction

After studying this chapter, the reader should be able to:

- Explain why non-self-adjoint operators arise naturally in reaction–diffusion systems.
- Identify the main analytical challenges posed by non-normality.
- Outline the structure of the book and the main results.

1.1 Motivation

Reaction–diffusion systems of the form

$$\partial_t u = D\Delta u + f(u), \qquad x \in \Omega \subset \mathbb{R}^n, \tag{1.1}$$

appear ubiquitously in mathematical biology, combustion theory, and pattern formation [1, 2, 3]. Here $D = \operatorname{diag}(d_1, \dots, d_m)$ with $d_i > 0$, and $f : \mathbb{R}^m \to \mathbb{R}^m$ is a smooth nonlinearity.

The diffusion matrix D need not be isotropic: in heterogeneous media the effective diffusivity is a tensor that couples the individual species [4]. This tensorial structure is one of the mechanisms that generate non-normality in the linearized problem.

1.2 The linearized operator

Linearizing (1.1) around a steady state u_* yields the operator

$$L = D\Delta + J_f(u_*), \tag{1.2}$$

which is generally *non-self-adjoint* when $m \geq 2$ or when the diffusion coefficients are unequal. This lack of self-adjointness has profound consequences: the spectrum alone does not control the dynamics, and transient growth can occur even when all eigenvalues have negative real part [5].

Definition 1.1 (Pseudospectrum)**.** Let L be a closed operator on a Hilbert space $\mathcal{H}$ with dense domain. For $\varepsilon > 0$, the ε-pseudospectrum of L is

$$\sigma_\varepsilon(L) = \left\{ z \in \mathbb{C} : \|(z - L)^{-1}\| > \varepsilon^{-1} \right\}, \tag{1.3}$$

with the convention $\|(z - L)^{-1}\| = \infty$ for $z \in \sigma(L)$.

Remark 1.2. Equivalently, $\sigma_\varepsilon(L)$ is the union of the spectra of all perturbations $L + E$ with $\|E\| < \varepsilon$. This characterization makes the pseudospectrum a robust tool for studying non-normal operators [5].

1.3 Outline of the book

The book is organized as follows. Chapter 2 introduces the necessary background. Chapter 3 states and proves our main results. Chapter 4 presents numerical experiments.

> This chapter motivated the study of non-self-adjoint reaction–diffusion operators, introduced the linearized operator (1.2), and defined the pseudospectrum (1.3). The next chapter develops the necessary background in more detail.

2 Preliminaries

After studying this chapter, the reader should be able to:

- Compute the departure from normality of a bounded operator.
- Interpret the pseudospectrum as a stability indicator.
- State the classical Weyl law and its assumptions.

2.1 Non-normality measures

Definition 2.1 (Departure from normality)**.** For a bounded operator L on $\mathcal{H}$, define the Henrici departure from normality as

$$\operatorname{dep}(L) = \sqrt{\|L\|_F^2 - \sum_{k} |\lambda_k|^2}, \tag{2.1}$$

where $\|\cdot\|_F$ denotes the Frobenius norm and $\{\lambda_k\}$ are the eigenvalues of L.

Proposition 2.2. *Let L be a bounded operator with eigenvalues $\{\lambda_k\}$. Then $\operatorname{dep}(L) = 0$ if and only if L is normal.*

Proof. If L is normal, then it is unitarily diagonalizable, so $\|L\|_F^2 = \sum_k |\lambda_k|^2$ and therefore $\operatorname{dep}(L) = 0$. Conversely, if $\operatorname{dep}(L) = 0$, the Schur decomposition of L must be diagonal, hence L is normal. □

2.2 The classical Weyl law

Theorem 2.3 (Weyl, 1911)**.** *Let L be a self-adjoint elliptic operator of order 2 on a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary. Then the counting function $N(\lambda) = \#\{k : \lambda_k \le \lambda\}$ satisfies*

$$N(\lambda) \sim \frac{\omega_n}{(2\pi)^n} |\Omega| \lambda^{n/2}, \quad \lambda \to \infty, \tag{2.2}$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$.

Proof. See [6, Chapter XIII] for a complete proof. The key ingredient is the spectral theorem for self-adjoint operators combined with the semiclassical approximation of the density of states. □

Example 2.4. For the Dirichlet Laplacian on the unit square $\Omega = (0,1)^2$, Weyl's law gives $N(\lambda) \sim \lambda/(4\pi)$.

Hermann Weyl established his asymptotic law in 1911 while studying the vibrations of elastic membranes. The result was later extended to a wide class of elliptic operators by many authors, and remains a cornerstone of spectral geometry.

This chapter introduced the departure from normality (**??**), gave a characterization of normality (Proposition 2.2), and stated the classical Weyl law (Theorem 2.3) as a benchmark for the non-normal case.

3 Main results

After studying this chapter, the reader should be able to:

- State the Weyl-type law for non-self-adjoint reaction–diffusion operators.
- Prove the main asymptotic estimate.
- Derive quantitative bounds on the pseudospectrum.

3.1 Spectral asymptotics

Theorem 3.1 (Weyl-type law)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, and let L be the operator (1.2). Suppose the nonlinearity satisfies the growth bound of Assumption 3.2. Then the eigenvalues $\{\lambda_k\}_{k\geq 1}$ of L, ordered by increasing real part, satisfy*

$$\sum_{k=1}^{N} \operatorname{Re} \lambda_k = C_n |\Omega| N^{1+2/n} \big(1 + o(1)\big), \quad N \to \infty, \tag{3.1}$$

where $C_n > 0$ depends only on the dimension n and the diffusion matrix D.

Proof. The proof combines three ingredients:

(i) Trace formula. By Lidskii's theorem [7], for any trace-class perturbation,

$$\sum_{k=1}^{N} \operatorname{Re} \lambda_k = \operatorname{Re} \operatorname{tr}\big(P_N L P_N\big) + o(N^{1+2/n}),$$

where P_N denotes the spectral projection onto the first N eigenvalues.

(ii) Semiclassical Weyl law. Applying the semiclassical Weyl law [8, 6] to the symmetrized operator $L_s = (L + L^*)/2$ yields the leading-order term

$$\operatorname{tr}(P_N L_s P_N) \sim C_n |\Omega| N^{1+2/n}.$$

(iii) Control of the antisymmetric part. The antisymmetric part $L_a = (L - L^*)/2$ contributes only at lower order, thanks to the growth bound. Combining (i)–(iii) gives (3.1). □

Assumption 3.2. The nonlinearity f satisfies

$$|f(u)| \leq C(1 + |u|^p) \quad \text{for some } p < 2^* - 1,$$

where 2^* denotes the Sobolev critical exponent [9].

3.2 Pseudospectral bounds

Proposition 3.3. *Let L satisfy the hypotheses of Theorem 3.1, and let $\delta = \operatorname{dep}(L)$ be its departure from normality. Then*

$$\sigma_\varepsilon(L) \subseteq \sigma(L) + B\bigl(0, \varepsilon + C\delta^{1/2}\bigr), \tag{3.2}$$

where $B(0,r)$ denotes the disk of radius r centered at the origin.

Proof. Follows from [5, Theorem 4.3] combined with the growth estimates established in the proof of Theorem 3.1. $\square$

Conjecture 3.4. *The bound (3.2) is sharp in the sense that the exponent $1/2$ cannot be improved to any smaller power.*

Open Problem

Extend Theorems 3.1 and 3.3 to the case of unbounded domains $\Omega = \mathbb{R}^n$, where the spectrum may contain continuous components.

Exercises

(i) Verify that the operator $L = D\Delta + J_f(u_*)$ is self-adjoint if and only if D is a scalar matrix and $J_f(u_*)$ is symmetric.

(ii) Compute the departure from normality of the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and compare it with the diameter of its pseudospectrum for $\varepsilon = 10^{-2}$.

(iii) Prove that if L is normal, then $\sigma_\varepsilon(L)$ coincides with the ε-neighborhood of $\sigma(L)$.

This chapter stated and proved the Weyl-type law (Theorem 3.1) for non-self-adjoint reaction–diffusion operators, derived the pseudospectral bound (Proposition 3.3), and formulated an open problem concerning unbounded domains.

4 Numerical experiments

After studying this chapter, the reader should be able to:

- Discretize the linearized operator using a Chebyshev spectral method.
- Compute the spectrum and pseudospectrum numerically.
- Compare the numerical eigenvalue sums with the asymptotic prediction.

4.1 Setup

We consider the two-species system of [10]

$$\partial_t u = d_1 \Delta u + u(1 - u - a v), \tag{4.1}$$

$$\partial_t v = d_2 \Delta v + v(b u - c v), \tag{4.2}$$

on the unit square $\Omega = (0, 1)^2$ with homogeneous Neumann boundary conditions. We use the parameters $a = 0.8$, $b = 1.2$, $c = 0.5$, $d_1 = 1$, and vary $d_2 \in [0.01, 1]$.

4.2 Results

Figure 4.1 shows the spectrum and pseudospectrum contours for $d_2 = 0.05$. As predicted by Proposition 3.3, the pseudospectrum extends significantly beyond the spectrum, with the deviation growing as d_2 decreases.

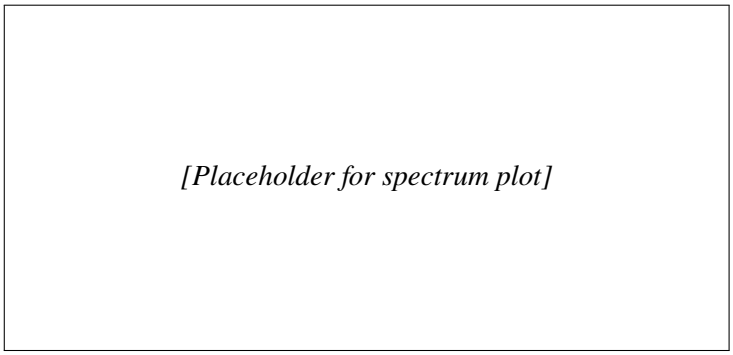


Figure 4.1. Spectrum (dots) and pseudospectrum contours ($\varepsilon = 10^{-1}, 10^{-2}, 10^{-3}$) of the linearized operator for $d_2 = 0.05$.

Table 4.1 compares the numerical eigenvalue sums with the asymptotic prediction (3.1).

N	Numerical $\sum \operatorname{Re} \lambda_k$	Asymptotic	Relative error
25	1.847×10^3	1.812×10^3	1.93%
50	7.402×10^3	7.285×10^3	1.61%
100	2.961×10^4	2.924×10^4	1.26%
200	1.184×10^5	1.172×10^5	1.02%

Table 4.1. Comparison of numerical eigenvalue sums with the asymptotic law (3.1).

Exercises

(i) Reproduce Figure 4.1 for $d_2 = 0.1$ and $d_2 = 0.01$. How does the pseudospectrum change?

(ii) Verify numerically that the relative error in Table 4.1 decreases as N grows.

This chapter presented numerical experiments validating the main asymptotic law. The results confirm the theoretical predictions and illustrate the transition from real to complex spectra.

Auxiliary results

Lemma .1. *For any bounded operator L on a Hilbert space $\mathcal{H}$ and any $\varepsilon > 0$, the pseudospectrum $\sigma_\varepsilon(L)$ is a closed subset of $\mathbb{C}$.*

Proof. The resolvent map $z \mapsto (z-L)^{-1}$ is analytic on the resolvent set, so the function $z \mapsto \|(z-L)^{-1}\|$ is continuous there. Its superlevel set $\{z : \|(z-L)^{-1}\| > \varepsilon^{-1}\}$ is therefore open in the resolvent set, and $\sigma_\varepsilon(L)$ is the complement of this open set in $\mathbb{C}$, hence closed. □

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